PERGAMON

International Journal of Solids and Structures 36 (1999) 2807–2824

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Transient response analysis of one-dimensional distributed parameter systems

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Received 6 May 1997; in revised form 27 March 1998

Abstract

A semi-analytic method is presented for the analysis of transient response of one-dimensional distributed parameter systems. Replacing time differentials by finite difference, the governing partial differential equations are reduced to difference–differential equations. The solutions of derived ordinary differential equations are given in exact and closed form by distributed transfer function method. Complex systems that contain many one-dimensional sub-systems are also studied. Numerical results show that the efficiency and accuracy of the method are excellent. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Distributed parameter dynamic systems that are composed of multiple one-dimensional sub-systems, such as various frame structures, bridge structures and pipe systems, are widely used in engineering. Their basic sub-systems are one-dimensional in space. The static response analysis of these problems is mature. There are many methods and commercial software available to analyze the problems and generally the solutions are accurate enough. The solutions of transient response problems are generally analyzed by numerical methods, such as finite element method, and analytic solutions are possible only in very few simple cases. However, numerical methods are at the cost of computer memory and CPU time and are inefficient for some cases, such as high frequency response.

Recently, Yang and Tan (1992), Yang (1994), Tan and Chung (1993) have published a series of research work on one-dimensional distributed parameter systems. Applying Laplace transform to time t, they reduced the partial differential equations of a one-dimensional dynamic system into ordinary differential equations with one spatial variable and one complex parameter. The solutions are given by transfer function method (TFM) in frequency domain. They also presented the

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PII: S0020–7683(98)00132–2

method for the analysis of complex systems composed of multiple one-dimensional sub-systems. For static and frequency responses, their method gives exact and closed form solutions.

Zhou and Yang (1995, 1996) and Yang and Zhou (1995, 1996) further extended the transfer function method to the analysis of two-dimensional problems. For cylindrical shells, they expanded the displacements in Fourier's series along circumferential direction to reduce the original problems into the solutions of a series of decoupled differential equations that can be solved by transfer function method. For plane deformation and plate bending, they proposed the strip distributed transfer function method (SDTFM) in which numerical discretization is introduced along one spatial coordinate direction. However, for transient response, the applications of TFM and SDTFM are difficult because both methods ask to find the inverse Laplace transforms of the solutions in state space.

In this paper, a semi-analytical method is presented for the analysis of transient response of one-dimensional distributed parameter systems. The basic idea of the method is replacing time differentials by finite difference to reduce the original partial differential equations into difference–differential equations that can be solved by state space technique in exact and closed form. The main advantages of the method are threefold. First, it avoids the difficulties in finding inverse Laplace transform of the transfer function solution. Second, the exact and closed form solution on spatial variable reduces the error introduced by numerical discretization so that the method has very high precision. Third, the method needs much less elements than that of FEM and the computational efficiency is higher. Moreover, the computer coding is easy for the method and it is possible to treat different problems in a systematic way like FEM in programming.

2. Basic formulations

In this section, a one-dimensional sub-system is first studied. The sub-system is supposed to be homogeneous so that the partial differential equations are of constant coefficients.

The governing dynamic equations of a one-dimensional distributed parameter system are

$$\sum_{k=1}^{n}\sum_{i=0}^{n_k}\left(a_{mki}+b_{mki}\frac{\partial}{\partial t}+c_{mki}\frac{\partial^2}{\partial t^2}\right)\frac{\partial^i u_k(x,t)}{\partial x^i}=f_m(x,t)\quad (m=1,2,\ldots,n) \tag{1}$$

where $u_k(x,t)$ $(k=1,2,\ldots,n)$ are field functions, such as displacements, forces, temperature, etc. n_k is the highest differential order of $u_k(x,t)$ with spatial coordinate x, a_{mki}, b_{mki}, and c_{mki} are constants, $f_m(x,t)$ $(m=1,2,\ldots,n)$ are external excitations.

Differential eqns (1) are subjected to following initial and boundary restrictions:

$$u_k(x,0)=\bar{u}_k(x),\left.\frac{\partial u_k}{\partial t}\right|_{t=0}=\bar{v}_k(x)\quad (k=1,2,\ldots,n) \tag{2}$$

$$\sum_{k=1}^{n}\sum_{i=0}^{n_k-1}\left.\left(\alpha_{lki}^{(j)}+\beta_{lki}^{(j)}\frac{\partial}{\partial t}+\chi_{lki}^{(j)}\frac{\partial^2}{\partial t^2}\right)\frac{\partial^i u_k(x,t)}{\partial x^i}\right|_{x=x_j}=\bar{g}_l(t)\quad (j=1,2;l=1,2,\ldots,N_j) \tag{3}$$

Where $\bar{u}_k(x)$, $\bar{v}_k(x)$ and $\bar{g}_l(t)$ are prescribed functions, $a_{lki}^{(j)}$, $\beta_{lki}^{(j)}$ and $\chi_{lki}^{(j)}$ are constants, x_j $(j=1,2)$ are the ends of the sub-system, and $N_1+N_2=N=\Sigma_{k=1}^{n} n_k$.

Define vector Φ, state space vector $\eta(x,t)$ and its sub-vector $\eta_k(x,t)$ as

$$\Phi = \left\{ \frac{\partial^{n_1} u_1}{\partial x^{n_1}} \frac{\partial^{n_2} u_2}{\partial x^{n_2}} \cdots \frac{\partial^{n_n} u_n}{\partial x^{n_n}} \right\}^T \in C^n \tag{4a}$$

$$\eta = \{\eta_1^T \quad \eta_2^T \quad \ldots \quad \eta_n^T\}^T \in C^N \tag{4b}$$

$$\eta_k(x,t) = \left\{ u_k(x,t) \frac{\partial u_k(x,t)}{\partial x} \cdots \frac{\partial^i u_k(x,t)}{\partial x^i} \cdots \frac{\partial^{n_k-1} u_k(x,t)}{\partial x^{n_k-1}} \right\}^T \in C^{n_k} \quad (k = 1, 2, \ldots, n) \tag{4c}$$

Equation (1) are cast into matrix form

$$\left(\bar{\mathbf{A}} + \bar{\mathbf{B}} \frac{\partial}{\partial t} + \bar{\mathbf{C}} \frac{\partial^2}{\partial t^2} \right) \cdot \Phi + \left(\mathbf{A} + \mathbf{B} \frac{\partial}{\partial t} + \mathbf{C} \frac{\partial^2}{\partial t^2} \right) \cdot \eta = \mathbf{f} \tag{5}$$

where

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{11n_1} & a_{12n_2} & \ldots & a_{1nn_n} \\ a_{21n_1} & a_{22n_2} & \ldots & a_{2nn_n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1n_1} & a_{n2n_2} & \ldots & a_{nnn_n} \end{bmatrix} \in C^{n \times n}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \ldots & \mathbf{A}_{1n} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \ldots & \mathbf{A}_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \ldots & \mathbf{A}_{nn} \end{bmatrix} \in C^{n \times N}$$

$$\mathbf{A}_{ij} = [a_{ij0} \quad a_{ij1} \quad \ldots \quad a_{ij(n_j-1)}] \in C^{1 \times n_j}$$

$\bar{\mathbf{B}}$ and $\mathbf{B}$, $\bar{\mathbf{C}}$ and $\mathbf{C}$ have the same forms as that of $\bar{\mathbf{A}}$ and $\mathbf{A}$ except that 'a_{ijp}' are replaced by 'b_{ijp}' and 'c_{ijp}' in $\bar{\mathbf{A}}$ and $\mathbf{A}_{ij}$, respectively.

Similarly, boundary conditions (3) are cast into

$$\left(\mathbf{M}^{(0)} + \mathbf{M}^{(1)} \frac{\partial}{\partial t} + \mathbf{M}^{(2)} \frac{\partial^2}{\partial t^2} \right) \cdot \eta(x_1, t) + \left(\mathbf{N}^{(0)} + \mathbf{N}^{(1)} \frac{\partial}{\partial t} + \mathbf{N}^{(2)} \frac{\partial^2}{\partial t^2} \right) \cdot \eta(x_2, t) = \gamma \tag{6}$$

in which

$$\mathbf{M}^{(0)} = \begin{bmatrix} \bar{\mathbf{M}}^{(0)} \\ \mathbf{0}_{N_2 \times N} \end{bmatrix} \in C^{N \times N}, \quad \mathbf{N}^{(0)} = \begin{bmatrix} \mathbf{0}_{N_1 \times N} \\ \bar{\mathbf{N}}^{(0)} \end{bmatrix} \in C^{N \times N}$$

$$\bar{\mathbf{M}}^{(0)} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \ldots & \mathbf{M}_{1n} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \ldots & \mathbf{M}_{n} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{M}_{N_1 1} & \mathbf{M}_{N_1 2} & \ldots & \mathbf{M}_{N_1 n} \end{bmatrix} \in C^{N_1 \times N},$$

$$\bar{\mathbf{N}}^{(0)} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & \dots & \mathbf{N}_{1n} \\ \mathbf{N}_{21} & \mathbf{N}_{22} & \dots & \mathbf{N}_{2N} \\ \dots & \dots & \dots & \dots \\ \mathbf{N}_{N_2 1} & \mathbf{N}_{N_2 2} & \dots & \mathbf{N}_{N_2 n} \end{bmatrix} \in C^{N_2 \times N}$$

$$\mathbf{M}_{ij} = [\alpha_{ij0}^{(1)} \quad \alpha_{ij1}^{(1)} \quad \dots \quad \alpha_{ij(n_j-1)}^{(1)}] \in C^{1 \times n_j}, \quad (i = 1, 2, \dots, N_1; j = 1, 2, \dots, n)$$

$$\mathbf{N}_{ij} = [\alpha_{ij0}^{(2)} \quad \alpha_{ij1}^{(2)} \quad \dots \quad \alpha_{ij(n_j-1)}^{(2)}] \in C^{1 \times n_j}, \quad (i = 1, 2, \dots, N_2; j = 1, 2, \dots, n)$$

with $\mathbf{0}_{k \times m}$ being $k \times m$ zero matrix. $\mathbf{M}^{(1)}$ and $\mathbf{N}^{(1)}$, $\mathbf{M}^{(2)}$ and $\mathbf{N}^{(2)}$ have the same definitions as that of $\mathbf{M}^{(0)}$ and $\mathbf{N}^{(0)}$ except $\alpha_{ijp}^{(k)}$ are replaced by $\beta_{ijp}^{(k)}$ and $\chi_{ijp}^{(k)}$ in $\mathbf{M}_{ij}$ and $\mathbf{N}_{ij}$, respectively.

Using the difference formulas introduced by Hilber and Hughes (1978) and Hughes (1983)

$$a_{t+\theta\Delta t} = (1-\theta)a_t + \theta a_{t+\Delta t} \tag{7a}$$

$$v_{t+\theta\Delta t} = v_t + \theta\Delta t\{(1-\gamma)a_t + \gamma a_{t+\theta\Delta t}\} \tag{7b}$$

$$d_{t+\theta\Delta t} = d_t + \theta\Delta t v_t + \frac{(\theta\Delta t)^2}{2}\{(1-2\beta)a_t + 2\beta a_{t+\theta\Delta t}\} \tag{7c}$$

$$d_{t+\Delta t} = d_t + \Delta t v_t + \frac{(\Delta t)^2}{2}\{(1-2\beta)a_t + 2\beta a_{t+\Delta t}\} \tag{7d}$$

$$v_{t+\Delta t} = v_t + \Delta t\{(1-\gamma)a_t + \gamma a_{t+\Delta t}\}, \tag{7e}$$

where $a = (\partial^2 u/\partial t^2)$, $v = (\partial u/\partial t)$, $d = u$, the subscript 't' denotes that the function takes its value at time t, so do the other subscripts in (7). θ, γ, β are adjustable parameters. Taking $\theta = 1$, (7) is corresponding to Newmark's method, taking $\beta = \frac{1}{6}$ and $\gamma = \frac{1}{2}$, it is the Wilson-θ method.

Solving $a_{t+\theta\Delta t}$ from (7c) and substituting it into (7b), we have

$$v_{t+\theta\Delta t} = -\frac{\gamma}{\beta\theta\Delta t}d_t + \left(1+\frac{\gamma}{\beta}\right)v_t + \left(\theta\Delta t - \frac{\theta\gamma\Delta t}{2\beta}\right)a_t + \frac{\gamma}{\beta\theta\Delta t}d_{t+\theta\Delta t} \tag{8a}$$

$$a_{t+\theta\Delta t} = -\frac{1}{\beta(\theta\Delta t)^2}d_t - \frac{1}{\beta\theta\Delta t}v_t + \left(1+\frac{1}{2\beta}\right)a_t + \frac{1}{\beta(\theta\Delta t)^2}d_{t+\theta\Delta t} \tag{8b}$$

The external excitations are also supposed to be linear for $t \in (t, t+\theta\Delta t)$

$$f_{t+\theta\Delta t} = f_t + \theta(f_{t+\Delta t} - f_t) \tag{9}$$

Replacing time differentials in (5) by finite difference formulas of (8) and solving Φ, the following difference–differential equations are derived

$$\Phi_{t+\theta\Delta t} = \mathbf{D}\cdot\eta_{t+\theta\Delta t} + \mathbf{D}^{(0)}\cdot\eta_t + \mathbf{D}^{(1)}\cdot\dot{\eta}_t + \mathbf{D}^{(2)}\cdot\ddot{\eta}_t + \mathbf{E}^{(0)}\cdot\Phi_t + \mathbf{E}^{(1)}\cdot\dot{\Phi}_t + \mathbf{E}^{(2)}\cdot\ddot{\Phi}_t + \bar{\mathbf{f}}_t + \bar{\mathbf{f}}_{t+\Delta t} \tag{10}$$

in which $(\dot{\ }) = (\partial/\partial t)$, $(\ddot{\ }) = (\partial^2/\partial t^2)$, and

$$\mathbf{D} = -\bar{\mathbf{D}}\cdot\left(\mathbf{A}+\frac{\gamma}{\beta\theta\Delta t}\mathbf{B}+\frac{1}{\beta(\theta\Delta t)^2}\mathbf{C}\right),\quad \mathbf{D}^{(0)} = \bar{\mathbf{D}}\cdot\left(\frac{\gamma}{\beta\theta\Delta t}\mathbf{B}+\frac{1}{\beta(\theta\Delta t)^2}\mathbf{C}\right)$$

$$\mathbf{D}^{(1)} = \bar{\mathbf{D}}\cdot\left(-\left(1-\frac{\gamma}{\beta}\right)\mathbf{B}+\frac{1}{\beta\theta\Delta t}\mathbf{C}\right),\quad \mathbf{D}^{(2)} = -\bar{\mathbf{D}}\cdot\left(\theta\Delta t\left(1-\frac{\gamma}{2\beta}\mathbf{B}+\left(1-\frac{1}{2\beta}\mathbf{C}\right)\right.\right.$$

$$\mathbf{E}^{(0)} = \bar{\mathbf{D}}\cdot\left(\frac{\gamma}{\beta\theta\Delta t}\bar{\mathbf{B}}+\frac{1}{\beta(\theta\Delta t)^2}\mathbf{C}\right),\quad \mathbf{E}^{(1)} = \bar{\mathbf{D}}\cdot\left(-\left(1-\frac{\gamma}{\beta}\right)\bar{\mathbf{B}}+\frac{1}{\beta\theta\Delta t}\bar{\mathbf{C}}\right),$$

$$\mathbf{E}^{(2)} = -\bar{\mathbf{D}}\cdot\left(\theta\Delta t\left(1-\frac{\gamma}{2\beta}\right)\bar{\mathbf{B}}+\left(1-\frac{1}{2\beta}\right)\mathbf{C}\right),\quad \bar{\mathbf{D}} = \left(\bar{\mathbf{A}}+\frac{\gamma}{\beta\theta\Delta t}\bar{\mathbf{B}}+\frac{1}{\beta(\theta\Delta t)^2}\bar{\mathbf{C}}\right)^{-1}$$

$$\bar{\mathbf{f}}_t = (1-\theta)\bar{\mathbf{D}}\cdot\mathbf{f}_t,\quad \bar{\mathbf{f}}_{t+\Delta t} = \theta\bar{\mathbf{D}}\cdot\mathbf{f}_{t+\Delta t}$$

Equation (10) is cast into state space form

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta_{t+\theta\Delta t} = \mathbf{F}\cdot\eta_{t+\theta\Delta t}+\mathbf{q}_{t+\theta\Delta t} \tag{11}$$

in which

$$\mathbf{F} = [\mathbf{F}_1^T \quad \mathbf{F}_2^T \quad \ldots \quad \mathbf{F}_i^T \quad \ldots \quad \mathbf{F}_n^T]^T \in C^{N\times N},\quad \mathbf{F}_i = \begin{bmatrix}\mathbf{F}_i^{(1)}\\ \mathbf{F}_i^{(2)}\end{bmatrix}\in C^{n_i\times N}$$

$$\mathbf{F}_i^{(1)} = [\mathbf{0}_{(n_i-1)\times N_i^{(1)}} \quad \mathbf{I}_{(n_i-1)\times(n_i-1)} \quad \mathbf{0}_{(n_i-1)\times N_i^{(2)}}],\quad \mathbf{F}_i^{(2)} = [d_{i1} \quad d_{i2} \quad \ldots \quad d_{in}]$$

$$\bar{\mathbf{F}}^{(0)} = [\bar{\mathbf{F}}_1^T \quad \bar{\mathbf{F}}_2^T \quad \ldots \quad \bar{\mathbf{F}}_i^T \quad \ldots \quad \bar{\mathbf{F}}_n^T]^T,\quad \bar{\mathbf{F}}_i = \begin{bmatrix}\mathbf{0}_{n_i\times n}\\ \mathbf{F}_i^{(2)}\end{bmatrix}$$

$$\mathbf{F}_i^{(2)} = [\mathbf{F}_{i1}^{(2)} \quad \mathbf{F}_{i2}^{(2)} \quad \ldots \quad \mathbf{F}_{ik}^{(2)} \quad \ldots \quad \mathbf{F}_{in}^{(2)}],\quad \mathbf{F}_{ik}^{(2)} = [\mathbf{0}_{1\times(n_k-1)} \quad e_{ik}^{(0)}]$$

$$\mathbf{q}_{t+\theta\Delta t} = (\mathbf{F}^{(0)}\eta_t+\bar{\mathbf{F}}^{(0)}\eta_t')+(\mathbf{F}^{(1)}\dot{\eta}_t+\bar{\mathbf{F}}^{(1)}\dot{\eta}_t')+(\mathbf{F}^{(2)}\ddot{\eta}_t+\bar{\mathbf{F}}^{(2)}\ddot{\eta}_t')+\mathbf{q}_t+\mathbf{q}_{t+\Delta t}$$

$$\mathbf{q}_t = \{\bar{\mathbf{q}}_1^T \quad \bar{\mathbf{q}}_2^T \quad \ldots \quad \bar{\mathbf{q}}_k^T \quad \ldots \quad \bar{\mathbf{q}}_n^T\}^T,\quad \bar{\mathbf{q}}_k^T = \{\mathbf{0}_{1\times(n_k-1)} \quad f_{t,k}\}^T$$

where $\mathbf{I}_{k\times k}$ is a $k\times k$ unit matrix, $N_i^{(1)} = 1+\Sigma_{k=1}^{i-1} n_k$ and $N_i^{(2)} = \Sigma_{k=i+1}^{n} n_k$. $\mathbf{F}^{(0)}$, $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ are given by replacing d_{ij} by $d_{ij}^{(0)}$, $d_{ij}^{(1)}$ and $d_{ij}^{(2)}$ in $\mathbf{F}$, respectively; $\bar{\mathbf{F}}^{(1)}$ and $\bar{\mathbf{F}}^{(2)}$ are given by replacing $e_{ik}^{(0)}$ by $e_{ij}^{(1)}$ and $e_{ik}^{(2)}$ in $\bar{\mathbf{F}}^{(0)}$, respectively, and d_{ij}, $d_{ij}^{(0)}$, $d_{ij}^{(1)}$, $d_{ij}^{(2)}$, $e_{ij}^{(0)}$, $e_{ij}^{(1)}$, $d_{ij}^{(2)}$ are elements of $\mathbf{D}_{ij}$, $\mathbf{D}_{ij}^{(0)}$, $\mathbf{D}_{ij}^{(1)}$, $\mathbf{D}_{ij}^{(2)}$, $\mathbf{E}_{ij}^{(0)}$, $\mathbf{E}_{ij}^{(1)}$ and $\mathbf{E}_{ij}^{(2)}$, respectively. Replacing $f_{t,k}$ by $f_{t+\Delta t,k}$ in $\mathbf{q}_t$ gives $\mathbf{q}_{t+\Delta t}$.

Similarly, the difference equation of boundary condition (6) is

$$\mathbf{M}\cdot\eta_{t+\theta\Delta t}(x_1)+\mathbf{N}\cdot\eta_{t+\theta\Delta t}(x_2) = \bar{\gamma} \tag{12}$$

where

$$\mathbf{M} = \left(\mathbf{M}^{(0)}+\frac{\gamma}{\beta\theta\Delta t}\mathbf{M}^{(1)}+\frac{1}{\beta(\theta\Delta t)^2}\mathbf{M}^{(2)}\right),\quad \mathbf{N} = \left(\mathbf{N}^{(0)}+\frac{\gamma}{\beta\theta\Delta t}\mathbf{N}^{(1)}+\frac{1}{\beta(\theta\Delta t)^2}\mathbf{N}^{(2)}\right)$$

$$\bar{\gamma} = (1-\theta)\gamma_t + \theta\gamma_{t+\Delta t} + \bar{\mathbf{M}}^{(0)} \cdot \eta_t(x_1) + \bar{\mathbf{M}}^{(1)} \cdot \dot{\eta}_t(x_1)$$
$$+ \bar{\mathbf{M}}^{(2)} \cdot \ddot{\eta}_t(x_1) + \bar{\mathbf{N}}^{(0)} \cdot \eta_t(x_2) + \bar{\mathbf{N}}^{(1)} \cdot \dot{\eta}_t(x_2) + \bar{\mathbf{N}}^{(2)} \cdot \ddot{\eta}_t(x_2)$$

$$\bar{\mathbf{M}}^{(0)} = \left(\frac{\gamma}{\beta\theta\Delta t}\mathbf{M}^{(1)} + \frac{1}{\beta(\theta\Delta t)^2}\mathbf{M}^{(2)}\right), \quad \bar{\mathbf{M}}^{(1)} = \left(-\left(1-\frac{\gamma}{\beta}\right)\mathbf{M}^{(1)} + \frac{1}{\beta\theta\Delta t}\mathbf{M}^{(2)}\right),$$

$$\bar{\mathbf{M}}^{(2)} = -\left(\theta\Delta t\left(1-\frac{\gamma}{2\beta}\right)\mathbf{M}^{(1)} + \left(1-\frac{1}{2\beta}\right)\mathbf{M}^{(2)}\right), \quad \bar{\mathbf{N}}^{(0)} = \left(\frac{\gamma}{\beta\theta\Delta t}\mathbf{N}^{(1)} + \frac{1}{\beta(\theta\Delta t)^2}\mathbf{N}^{(2)}\right)$$

$$\bar{\mathbf{N}}^{(1)} = \left(-\left(1-\frac{\gamma}{\beta}\right)\mathbf{N}^{(1)} + \frac{1}{\beta\theta\Delta t}\mathbf{N}^{(2)}\right), \quad \bar{\mathbf{N}}^{(2)} = -\left(\theta\Delta t\left(1-\frac{\gamma}{2\beta}\right)\mathbf{N}^{(1)} + \left(1-\frac{1}{2\beta}\right)\mathbf{N}^{(2)}\right)$$

The solution of (11) under boundary restriction (12) is

$$\eta_{t+\theta\Delta t}(x) = \int_{x_1}^{x_2} \mathbf{G}(x,\xi) \cdot \mathbf{q}_{t+\theta\Delta t}(\xi)\,\mathrm{d}\xi + \mathbf{H}(x) \cdot \gamma_{t+\theta\Delta t} \tag{13}$$

in which

$$\mathbf{G}(x,\xi) = \begin{cases} \mathrm{e}^{\mathbf{F}x} \cdot [\mathbf{M} \cdot \mathrm{e}^{\mathbf{F}x_1} + \mathbf{N} \cdot \mathrm{e}^{\mathbf{F}x_2}]^{-1} \cdot \mathbf{M} \cdot \mathrm{e}^{\mathbf{F}(x_1-\xi)} & \xi < x \\ -\mathrm{e}^{\mathbf{F}x} \cdot [\mathbf{M} \cdot \mathrm{e}^{\mathbf{F}x_1} + \mathbf{N} \cdot \mathrm{e}^{\mathbf{F}x_2}]^{-1} \cdot \mathbf{N} \cdot \mathrm{e}^{\mathbf{F}(x_2-\xi)} & \xi > x \end{cases} \tag{14a}$$

$$\mathbf{H}(x) = \mathrm{e}^{\mathbf{F}x} \cdot [\mathbf{M} \cdot \mathrm{e}^{\mathbf{F}x_1} + \mathbf{N} \cdot \mathrm{e}^{\mathbf{F}x_2}]^{-1} \tag{14b}$$

Having $\eta_{t+\theta\Delta t}(x)$ known from (13), $\dot{\eta}_{t+\theta\Delta t}(x)$ and $\ddot{\eta}_{t+\theta\Delta t}(x)$ are calculated by the following formulas

$$\ddot{\eta}_{t+\theta\Delta t} = \frac{1}{\beta(\theta\Delta t)^2}(\eta_{t+\theta\Delta t} - \eta_t - \theta\Delta t\dot{\eta}_t) - \frac{1-2\beta}{2\beta}\ddot{\eta}_t \tag{15a}$$

$$\dot{\eta}_{t+\theta\Delta t} = \dot{\eta}_t + \theta\Delta t\{(1-\gamma)\ddot{\eta}_t + \gamma\ddot{\eta}_{t+\theta\Delta t}\} \tag{15b}$$

and the values of η at $t+\Delta t$ are given by

$$\ddot{\eta}_{t+\Delta t} = \frac{1}{\theta}\ddot{\eta}_{t+\theta\Delta t} + \left(1-\frac{1}{\theta}\right)\ddot{\eta}_t \tag{16a}$$

$$\dot{\eta}_{t+\Delta t} = \dot{\eta}_t + \Delta t\{(1-\gamma)\ddot{\eta}_t + \gamma\ddot{\eta}_{t+\Delta t}\} \tag{16b}$$

$$\eta_{t+\Delta t} = \eta_t + \Delta t\dot{\eta}_t + \frac{(\Delta t)^2}{2}\{(1-2\beta)\ddot{\eta}_t + 2\beta\ddot{\eta}_{t+\Delta t}\} \tag{16c}$$

Therefore, if η_t, $\dot{\eta}_t$, $\ddot{\eta}_t$ are known, $\eta_{t+\Delta t}$, $\dot{\eta}_{t+\Delta t}$, $\ddot{\eta}_{t+\Delta t}$ can be found from the above derived formulas.

Because $\mathbf{q}_{t+\theta\Delta t}$ of (11) is related to both the external loads and the transient response η_t, η'_t, $\dot{\eta}_t$, $\dot{\eta}'_t$, $\ddot{\eta}_t$, $\ddot{\eta}'_t$, the integral of (13) is not easy to be calculated exactly. Approximate integral is proposed for it. The region $[x_1, x_2]$ is divided into n sub-regions $[x^{(i)}, x^{(i+1)}]$ with $x_1 = x^{(1)} < x^{(2)} < \cdots < x^{(i)} < x^{(i+1)} < \cdots < x^{(n+1)} = x_2$. In each sub-region, η_t, η'_t, $\dot{\eta}_t$, $\dot{\eta}'_t$, $\ddot{\eta}_t$, $\ddot{\eta}'_t$ are interpolated

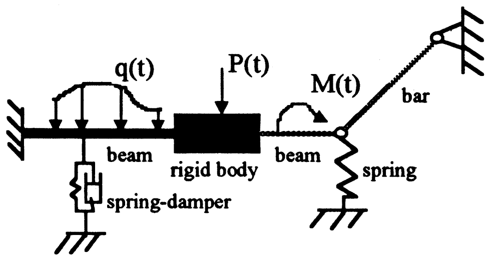

Fig. 1. A one-dimensional distributed parameter system.

by their values at $x^{(i)}$ $(i = 1, 2, \ldots, n+1)$ in low order polynomials. Then, the integral of (13) can be done with no difficulties. For the part of integral of (13) related to external loads, numerical approximation can also be used. The benefits of this approximation are the saving of CPU time and the possibility of computer codes for general purposes. In the numerical examples presented in the following sections, these approximate procedures are used.

3. Complex systems composed of multiple 1-dimensional sub-systems

Dynamic systems made of multiple one-dimensional sub-systems are studied in this section. Figure 1 is an example where the dynamic system includes two beams, a rigid body, a spring-damper, a spring and a bar. The formulas derived previously are valid for each one-dimensional sub-system. The ends of each one-dimensional sub-system are called nodes and the nodal parameter vector are defined as

$$\gamma = \begin{Bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{Bmatrix}, \quad \mathbf{d}_i = \begin{Bmatrix} \mathbf{d}_i^1(x_i) \\ \mathbf{d}_i^2(x_i) \\ \vdots \\ \mathbf{d}_i^n(x_i) \end{Bmatrix} \tag{17}$$

where $i = 1$ and 2,

$$\mathbf{d}_i^k(x_i, t) = \left\{ u_k(x_i, t) \frac{\partial u_k(x_i, t)}{\partial x} \cdots \frac{\partial^{p_k - 1} u_k(x_i, t)}{\partial x^{p_k - 1}} \right\}^T, \quad p_k = \left[\frac{n_k}{2} \right], \quad k = 1, 2, \ldots, n$$

$[y]$ denotes the integer part of y.

The boundary condition matrices of (12) are simplified to

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0}_{(N/2) \times N} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0}_{(N/2) \times N} \\ \mathbf{B} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \vdots \\ \mathbf{B}_n \end{bmatrix}$$

$$\mathbf{B}_k = [\mathbf{0}_{p_k \times r_k} \quad \mathbf{I}_{p_k \times p_k} \quad \mathbf{0}_{p_k \times s_k}]$$

with $r_k = \Sigma_{j=1}^{k-1} n_j$ and $s_k = p_k + \Sigma_{j=k+1}^{n} n_j$.

For a linear system, the generalized force vectors are related to state space vector $\eta(x, t)$ by following constitutive relation

$$\sigma(x, t) = \mathbf{S} \cdot \eta(x, t) \tag{18}$$

where $\mathbf{S}$ is the constitutive matrix.

At time $t + \theta\Delta t$, we have

$$\sigma_{t+\theta\Delta t} = \mathbf{S} \cdot \eta_{t+\theta\Delta t} \tag{19}$$

Plugging (13) into (19) gives

$$\sigma^{(i)}_{t+\theta\Delta t} = \mathbf{K}^{(i)} \cdot \gamma_{t+\theta\Delta t} + \mathbf{q}^{(i)} \quad (i = 1, 2) \tag{20}$$

where $i = 1$ and 2 are the ends of the sub-system, and

$$\mathbf{K}^{(i)} = \mathbf{S} \cdot \mathbf{H}(x_i)$$

$$\mathbf{q}^{(i)} = \mathbf{S} \cdot \int_{x_1}^{x_2} \mathbf{G}(x_i, \xi) \cdot \mathbf{q}_{t+\theta\Delta t}(\xi)\, \mathrm{d}\xi$$

At each node, assembling equilibrium equations according to equilibrium of forces, following simultaneous linear algebraic equations will be obtained

$$\mathbf{K} \cdot \mathbf{d} = \mathbf{Q} \tag{21}$$

in which $\mathbf{K}$, $\mathbf{d}$ and $\mathbf{Q}$ are the global stiffness matrix, nodal parameter vector and load vector, respectively. Applying proper boundary conditions, (21) is solved in the standard way to give the nodal parameter vector. Then, $\eta_{t+\theta\Delta t}(x)$ can be found from (13) for each subsystem.

4. Transient response of a simple Euler–Bernoulli beam

For a specific problem, the dynamic equations and boundary conditions are generally much simpler than that of Section 2. Therefore, more direct derivations will be used in the following two sections. Moreover, to make the statements simpler, the Wilson-θ method is used. The transient response of a simple Euler–Bernoulli beam is first analyzed to illustrate the efficiency of the present method. The governing equations are

$$\text{Dynamic equation:} \quad EI\frac{\partial^4 w}{\partial x^4} + c\frac{\partial w}{\partial t} + \rho A\frac{\partial^2 w}{\partial t^2} = f(x, t) \tag{22}$$

$$\text{Initial conditions:} \quad w(x, 0) = w_0(x), \left.\frac{\partial w}{\partial t}\right|_{t=0} = v_0(x) \tag{23}$$

Boundary conditions: (1) $w(x_i,t) = \tilde{w}_i(t)$ or $EI\dfrac{\partial^3 w}{\partial x^3}\bigg|_{x=x_i} = \tilde{Q}_i(t)$ (24a)

$$(2)\quad \frac{\partial w}{\partial x}\bigg|_{x=x_i} = \tilde{\beta}_i(t) \quad \text{or} \quad -EI\frac{\partial^2 w}{\partial x^2}\bigg|_{x=x_i} = \tilde{M}_i(t) \tag{24b}$$

in which EI is the bending stiffness, c is the damping coefficient, ρA is the mass density per unit length, $w(x,t)$ is the flexure, $f(x,t)$ is the external excitation, $w_0(x)$, $v_0(x)$, $\tilde{w}_i(t)$, $\tilde{Q}_i(t)$, $\tilde{\beta}_i(t)$ and $\tilde{M}_i(t)$ $(i = 1,2)$ are given functions.

Applying the Wilson-θ method to (22) gives the following difference–differential equations

$$EI\frac{\partial^4 w_{t+\theta\Delta t}}{\partial x^4} + \left(\frac{3c}{\theta\Delta t} + \frac{6\rho A}{\theta^2\Delta t^2}\right)w_{t+\theta\Delta t} = f_t + \theta(f_{t+\Delta t} - f_t)$$
$$+\left(\frac{3c}{\theta\Delta t} + \frac{6\rho A}{\theta^2\Delta t^2}\right)w_t + \left(2c + \frac{6\rho A}{\theta\Delta t}\right)\dot{w}_t + \left(\frac{1}{2}c\theta\Delta t + 2\rho A\right)\ddot{w}_t \tag{25}$$

The state space vector is defined as

$$\eta_{t+\theta\Delta t} = \left\{ w_{t+\theta\Delta t} \quad \frac{\mathrm{d}}{\mathrm{d}x}w_{t+\theta\Delta t} \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}w_{t+\theta\Delta t} \quad \frac{\mathrm{d}^3}{\mathrm{d}x^3}w_{t+\theta\Delta t} \right\}^T \tag{26}$$

and the state space form of (25) is

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta_{t+\theta\Delta t} = \mathbf{F}\cdot\eta_{t+\theta\Delta t} + \mathbf{q} \tag{27}$$

in which

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{1}{EI}\left(\dfrac{3c}{\theta\Delta t} + \dfrac{6\rho A}{\theta^2\Delta t^2}\right) & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{q} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ q_4 \end{Bmatrix}$$

and

$$q_4 = \frac{1}{EI}\left\{f_t + \theta(f_{t+\Delta t} - f_t) + \left(\frac{3c}{\theta\Delta t} + \frac{6\rho A}{\theta^2\Delta t^2}\right)w_t + \left(2c + \frac{6\rho A}{\theta\Delta t}\right)\dot{w}_t + \left(\frac{1}{2}c\theta\Delta t + 2\rho A\right)\ddot{w}_t\right\}$$

The matrix form of boundary conditions (24) is

$$\mathbf{M}\cdot\eta_{t+\theta\Delta t}(x_1) + \mathbf{N}\cdot\eta_{t+\theta\Delta t}(x_2) = \theta\gamma_{t+\Delta t} + (1-\theta)\gamma_t \tag{28}$$

with

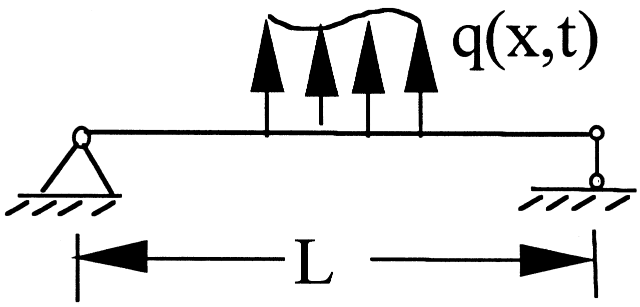

Fig. 2. Simply supported beam subjected to alternating load.

$$\mathbf{M} = \begin{bmatrix} \mathbf{B}^L \\ \mathbf{0}_{2\times 4} \end{bmatrix} \cdot \mathbf{B}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0}_{2\times 4} \\ \mathbf{B}^R \end{bmatrix} \cdot \mathbf{B}, \quad \gamma_{\bar{t}} = \begin{Bmatrix} \mathbf{B}^L \cdot \tilde{\gamma}_{1,\bar{t}} \\ \mathbf{B}^R \cdot \tilde{\gamma}_{2,\bar{t}} \end{Bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & EI \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -EI & 0 \end{bmatrix}, \quad \tilde{\gamma}_{i,\bar{t}} = \begin{Bmatrix} \tilde{w}_{i,\bar{t}} \\ \tilde{Q}_{i,\bar{t}} \\ \tilde{\beta}_{i,\bar{t}} \\ \tilde{M}_{i,\bar{t}} \end{Bmatrix}$$

where subscript '$\bar{t}$' takes values t or $t+\Delta t$, $i = 1$ and 2, $\mathbf{B}^L$ and $\mathbf{B}^R$ are 2×4 matrices called the selective matrices of boundary conditions at the left and right ends of the beam, respectively. The ith row of $\mathbf{B}^L$ is corresponding to the ith boundary condition listed in (24). If the ith boundary condition at the left end is a displacement condition, then

$$B^L_{i,2i-1} = 1; \quad B^L_{i,j} = 0 \quad (j \neq 2i-1)$$

Otherwise

$$B^L_{i,2i} = 1; \quad B^L_{i,j} = 0 \quad (j \neq 2i)$$

$\mathbf{B}^R$ is similar to $\mathbf{B}^L$ except that it is related to the right end of the beam.

A simply supported beam subjected to an alternating load $q = q_0(x)\sin\omega t$ (see Fig. 2) is analyzed by the present method and numerical results are compared with that of the finite element method. The material and geometric parameters are $\rho A = 10$, $EI = 20$, $c = 15$, $L = 3$, and $\theta = 1.4$. Numerical results are given in Tables 1 and 2 where $q_0(x) = 1000\delta[x-(L/2)]$, $\omega = 38$ and 125, respectively. It is easy to see that our method is very accurate and efficient. Taking the whole beam as one element, its precision is better than that of the finite element method using 30 elements, and the CPU time of the method is only about 40% of that consumed by FEM. Along with the increment of frequency ω, the advantages of our method becomes more obvious. When $\omega \leqslant 38$, the maximum relative error of FEM using 30 elements is within 1%. When $\omega = 125$, it increases to 8%. However, for our method, this error keeps within 0.5%. From Table 3, in which the beam is subjected to the distributed load $q_0(x) = 1000(16/L^4)[x-(L/2)]^4$, the same conclusions can be drawn. Undoubtedly, this advantage is very useful, especially for high frequency response analysis, on-line control and real-time simulation.

Table 1
Middle point displacement of simply supported beam ($\omega = 38$, $\Delta t = 0.01$)

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	CPU time	Method
0.0803	−0.6011	−0.8154	−0.6028	−0.3039	416	This paper
0.0823	−0.5728	−0.7486	−0.4945	−0.1691	199	FEM10 elements
0.0803	−0.5996	−0.8123	−0.5947	−0.2972	527	FEM20 elements
0.0803	−0.6008	−0.8153	−0.5997	−0.3038	1068	FEM30 elements
0.0803	−0.6010	−0.8159	−0.6008	−0.3053	2513	FEM50 elements
0.0803	−0.6010	−0.8160	−0.6010	−0.3055	11,009	FEM100 elements

Table 2
Middle point displacement of simply supported beam ($\omega = 125$, $\Delta t = 0.005$)

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	CPU time	Method
0.1721	0.07126	0.02917	0.00690	0.00702	825	This paper
0.1722	0.09197	0.04402	0.00852	0.02104	324	FEM10 elements
0.1719	0.07198	0.04073	0.00741	0.00743	1001	FEM20 elements
0.1720	0.07125	0.03135	0.00707	0.00708	2041	FEM30 elements
0.1720	0.07109	0.02931	0.00601	0.00700	4991	FEM50 elements
0.1720	0.07107	0.02903	0.00688	0.00699	19,748	FEM100 elements

Table 3
Middle point displacement of simply supported beam ($\omega = 76$, $\Delta t = 0.008$)

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	CPU time	Method
0.107432	0.059553	0.021923	0.008241	−0.010123	535	This paper
0.130218	0.066556	0.027726	0.005888	−0.018346	344	FEM10 elements
0.108739	0.059931	0.022300	0.008192	−0.010521	972	FEM20 elements
0.107588	0.059593	0.021967	0.008239	−0.010173	1601	FEM30 elements
0.107338	0.059518	0.021896	0.008248	−0.010098	4070	FEM50 elements
0.107303	0.059507	0.021885	0.008249	−0.010087	12,481	FEM100 elements

5. Transient response of frame structures

To show the solution procedures and efficiency of the present method in the analysis of complex one-dimensional systems, frame structures made of simple elastic beams are studied here. The local

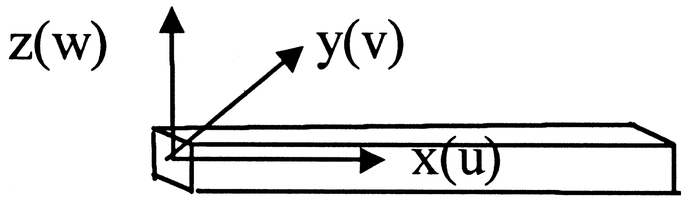

Fig. 3. Local coordinate system of the beam.

coordinate system of the beam is defined as shown in Fig. 3 where y and z are the principle axes of inertial of the cross-section of the beam. The dynamic equations are

$$EA\frac{\partial^2 u}{\partial x^2}-\rho A\frac{\partial^2 u}{\partial t^2}+f_x=0,\quad EI_z\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}-\rho I_z\frac{\partial^4 v}{\partial x^2\,\partial t^2}-f_y=0 \tag{29a,b}$$

$$EI_y\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I_y\frac{\partial^4 w}{\partial x^2\,\partial t^2}-f_t=0,\quad \alpha GI_x\frac{\partial^2 \varphi}{\partial x^2}-\rho I_x\frac{\partial^2 \varphi}{\partial t^2}+m_x=0 \tag{29c,d}$$

where u, v, w are the displacements in x-, y- and z-directions, φ is the torsion angle of the beam cross-section about x-axis, I_x, I_y, I_z are the rotation inertias about x, y and z axes. α is the torsion coefficient of the cross-section, f_x, f_y and f_z are distributed external excitations.

Because the differential equations in (29) are not coupled, to raise the computer efficiency, the state space vectors of the beam are defined as

$$\eta_{1,t+\theta\Delta t}=\left\{u_{t+\theta\Delta t}\ \frac{\mathrm{d}u_{t+\theta\Delta t}}{\mathrm{d}x}\right\}^T \tag{30a}$$

$$\eta_{2,t+\theta\Delta t}=\left\{v_{t+\theta\Delta t}\ \frac{\mathrm{d}v_{t+\theta\Delta t}}{\mathrm{d}x}\ \frac{\mathrm{d}^2v_{t+\theta\Delta t}}{\mathrm{d}x^2}\ \frac{\mathrm{d}^3v_{t+\theta\Delta t}}{\mathrm{d}x^3}\right\}^T \tag{30b}$$

$$\eta_{3,t+\theta\Delta t}=\left\{w_{t+\theta\Delta t}\ \frac{\mathrm{d}w_{t+\theta\Delta t}}{\mathrm{d}x}\ \frac{\mathrm{d}^2w_{t+\theta\Delta t}}{\mathrm{d}x^2}\ \frac{\mathrm{d}^3w_{t+\theta\Delta t}}{\mathrm{d}x^3}\right\}^T \tag{30c}$$

$$\eta_{4,t+\theta\Delta t}=\left\{\varphi_{t+\theta\Delta t}\ \frac{\mathrm{d}\varphi_{t+\theta\Delta t}}{\mathrm{d}x}\right\}^T \tag{30d}$$

and (29) are cast into the following state space forms

$$\frac{\mathrm{d}}{\mathrm{d}x}\eta_{i,t+\theta\Delta t}=\mathbf{F}_i\cdot\eta_{i,t+\theta\Delta t}+\mathbf{f}_i\quad (i=1,2,3,4) \tag{31}$$

where

$$\mathbf{F}_1=\begin{bmatrix}0 & 1\\ \dfrac{6\rho}{E\theta^2\Delta t^2} & 0\end{bmatrix},\quad \mathbf{f}_1=\begin{Bmatrix}0\\ f_1^*\end{Bmatrix}$$

$$\mathbf{F}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{6\rho A}{EI_z\theta^2\Delta t^2} & 0 & \dfrac{6\rho}{E\theta^2\Delta t^2} & 0 \end{bmatrix}, \quad \mathbf{f}_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f_2^* \end{Bmatrix}$$

$$\mathbf{F}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{6\rho A}{EI_y\theta^2\Delta t^2} & 0 & \dfrac{6\rho}{E\theta^2\Delta t^2} & 0 \end{bmatrix}, \quad \mathbf{f}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ f_3^* \end{Bmatrix}$$

$$\mathbf{F}_4 = \begin{bmatrix} 0 & 1 \\ \dfrac{6\rho}{\alpha G\theta^2\Delta t^2} & 0 \end{bmatrix}, \quad \mathbf{f}_4 = \begin{Bmatrix} 0 \\ f_4^* \end{Bmatrix}$$

and f_1^*, f_2^*, f_3^* and f_4^* have similar definitions as q_4.

The nodal displacement vectors of the beam are defined as

$$\gamma_1 = \{u_{t+\theta\Delta t}(x_1) \quad u_{t+\theta\Delta t}(x_2)\}^T \tag{32a}$$

$$\gamma_2 = \left\{ v_{t+\theta\Delta t}(x_1) \quad \frac{\mathrm{d}}{\mathrm{d}x} v_{t+\theta\Delta t}\bigg|_{x=x_1} \quad v_{t+\theta\Delta t}(x_2) \quad \frac{\mathrm{d}}{\mathrm{d}x} v_{t+\theta\Delta t}\bigg|_{x=x_2} \right\}^T \tag{32b}$$

$$\gamma_3 = \left\{ w_{t+\theta\Delta t}(x_1) \quad \frac{\mathrm{d}}{\mathrm{d}x} w_{t+\theta\Delta t}\bigg|_{x=x_1} \quad w_{t+\theta\Delta t}(x_2) \quad \frac{\mathrm{d}}{\mathrm{d}x} w_{t+\theta\Delta t}\bigg|_{x=x_2} \right\}^T \tag{32c}$$

$$\gamma_4 = \{\varphi_{t+\theta\Delta t}(x_1) \quad \varphi_{t+\theta\Delta t}(x_2)\}^T \tag{32d}$$

The boundary conditions are

$$\mathbf{M}_i \cdot \eta_{i,t+\theta\Delta t}(x_1) + \mathbf{N}_i \cdot \eta_{i,t+\theta\Delta t}(x_2) = \gamma_i \tag{33}$$

with

$$\mathbf{M}_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{N}_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ for } i = 1, 4 \text{ and}$$

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix}, \quad \mathbf{N}_i = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{I}_{2\times 2} & \mathbf{0}_{2\times 2} \end{bmatrix} \text{ for } i = 2 \text{ and } 3.$$

In the local coordinate system, the internal force vector is

$$\sigma = \{\sigma_1^T \quad \sigma_2^T \quad \sigma_3^T \quad \sigma_4^T\} \tag{34}$$

where

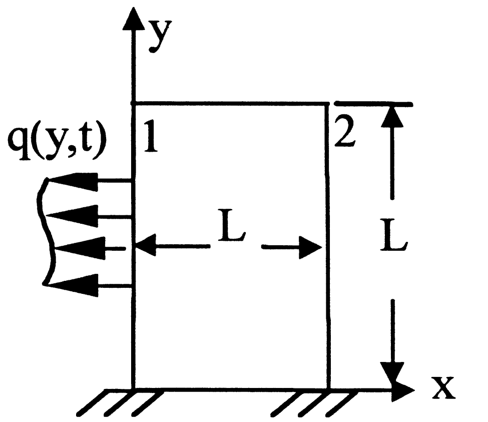

Fig. 4a. A plane frame subjected to in-plane alternating load.

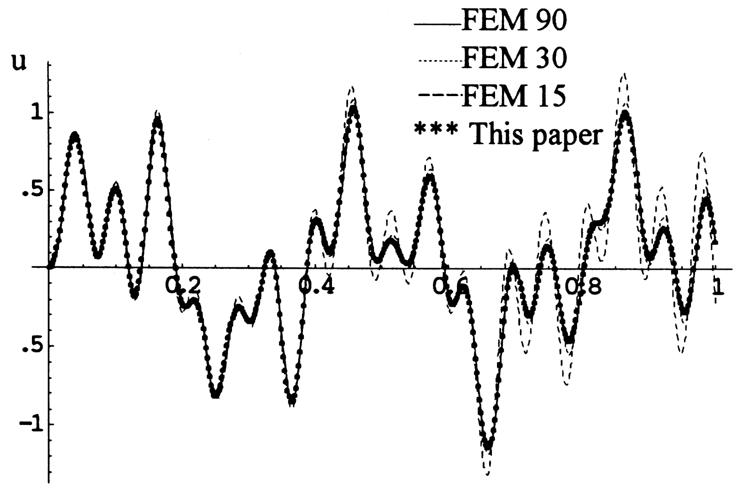

Fig. 4b. Transient response of node 1.

$$\sigma_i = \mathbf{S}_i \cdot \eta_i, \quad \mathbf{S}_1 = [0 \quad EA], \quad \mathbf{S}_4 = [0 \quad \alpha GI_x]$$

$$\mathbf{S}_2 = \begin{bmatrix} 0 & 0 & EI_z & 0 \\ 0 & 0 & 0 & -EI_z \end{bmatrix}, \quad \mathbf{S}_3 = \begin{bmatrix} 0 & 0 & EI_y & 0 \\ 0 & 0 & 0 & -EI_y \end{bmatrix}$$

Three different frame structures are analyzed to show the applications of the method. Each frame structure is taken to consist of simple beams with the same material and geometric parameters.

(1) Transient response of a plane frame structure subjected to in-plane alternating load. The plane frame structure is shown in Fig. 4a. The parameters used in calculation are $\rho A = 0.5$, $EI = 20$, $EA = 200$, $L = 1$. Numerical results for $q(y,t) = q_0(4/L^2)[y-(L/2)]^2 \sin \omega t$, in which $q_0 = 5000$

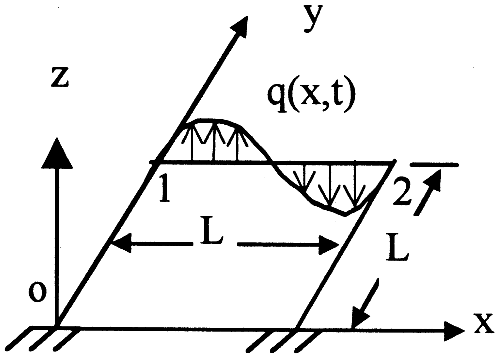

Fig. 5a. A plane frame subjected to out-of-plane loads.

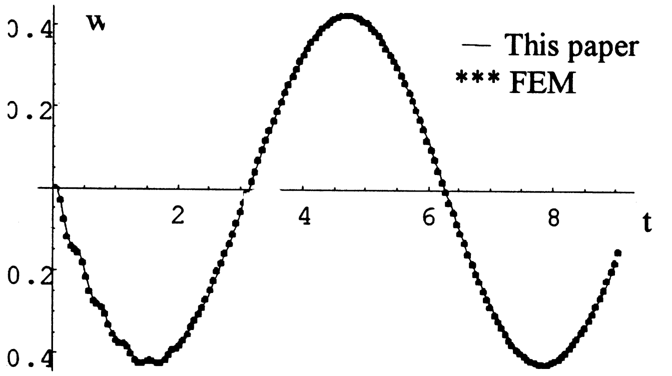

Fig. 5b. Transient response in z-direction of node 2.

and $\omega = 108$, are illustrated in Fig. 4b. As a comparison, finite element solutions using 15, 30 and 90 elements are also shown.

(2) Dynamic response of a plane frame subjected to out-of-plane alternating load. The plane frame structure is shown in Fig. 5a. The parameters used in numerical simulation are $E = 3 \times 10^7$, $G = 1.2 \times 10^7$, $\rho A = 10$, $L = 60$, $I_x = I_y = 100$, $\alpha I_x = 125$, $J = 200$. The distributed external load $q(x, t) = 5000 \sin(2\pi x/L) s\, \mathrm{in}\, t$. Transient responses of nodes 1 and 2 are illustrated in Fig. 5b and Fig. 5c in which FEM results are obtained using 30 elements.

(3) Space frame structure subjected to alternating load. The structure is shown in Fig. 6a. The material and geometric parameters are the same as that of example 2. $P_1 = 5000 \sin t$, $P_2 = 5000 \sin 3t$ and $P_3 = 5000 \sin 5t$. Transient responses of node A are illustrated in Figs 6b–d in which FEM results use 80 elements.

6. Conclusions

Theoretical analysis and numerical simulation show that the method presented in this paper has the following advantages:

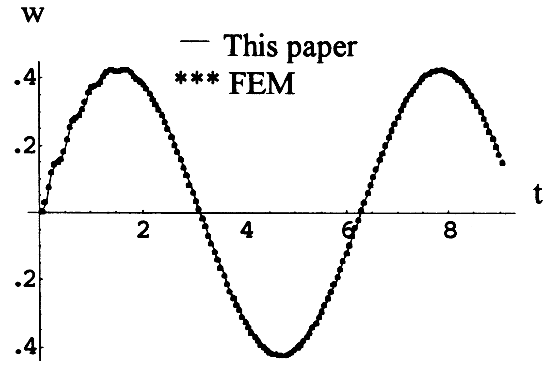

Fig. 5c. Transient response in z-direction of node 1.

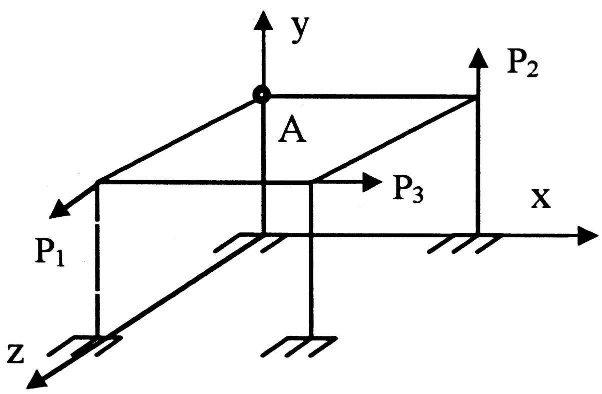

Fig. 6a. A space frame structure subjected to alternating loads.

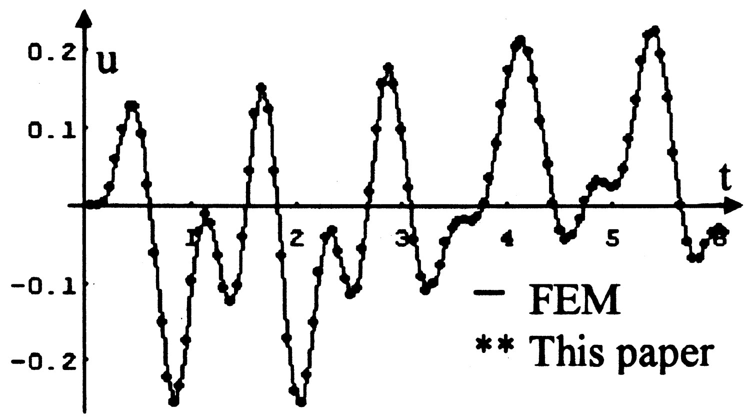

Fig. 6b. Transient response in x-direction of node A.

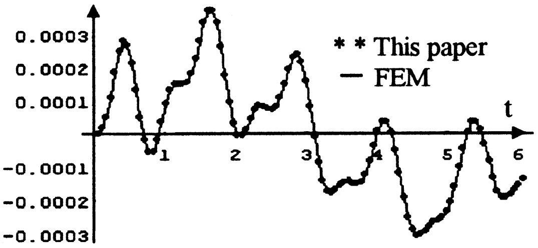

Fig. 6c. Transient response in y-direction of node A.

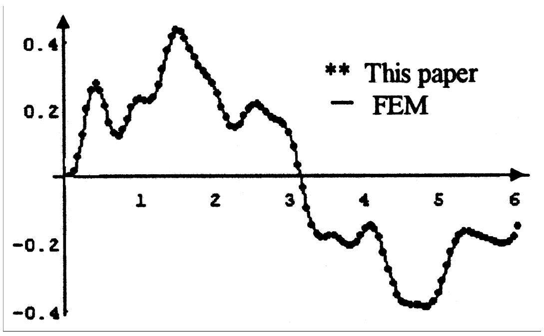

Fig. 6d. Transient response in z-direction of node A.

(1) It presents a new, semi-analytical method that avoids discretization of spatial coordinates. In the analysis of one-dimensional distributed parameter systems, the method takes a one-dimensional sub-system as one element and defines its end as nodes. Therefore, the scale of the global nodal equilibrium equations is much smaller than that of FEM, and the solution efficiency is much higher.
(2) The exact and closed form solution in spatial coordinates raises the precision and efficiency greatly. To find the transient response of a single beam, FEM has to use many elements, especially for high frequency response. However, taking a single beam as an element, the numerical error of our method is within 0.5% that is accurate enough for most engineering applications.
(3) The analytic procedure is standard and systematic for different physical and engineering problems. If the governing differential equations, initial and boundary conditions are known, the problem can be analyzed in a standard way. So does the computer coding. Therefore, the method will be attractive for commercial software.

Although the discussion is limited to differential equations within constant coefficients, the method presented in this paper can be extended to differential equations with variable coefficients in which the solution for spatial variables should be obtained by approximate methods.

Acknowledgements

Project 19572027 supported by the National Natural Science Foundation of China. Also supported by the National Committee of Education and National University of Defense Technology.

References

Chung, C.H., Tan, C.A., 1993. Transfer function formulation of constrained distributed parameter systems, Part II, applications. ASME J. Appl. Mech. 60, 1012–1019.

Hilber, H.M., Hughes, T.J.R., 1978. Earthquake Engineering and Structure Dynamics 16, 99–117.

Hughes, T.J.R., 1983. Computational Methods for Transient Analysis. Elsevier Science, B.V.

Tan, C.A., Chung, C.H., 1993. Transfer function formulation of constrained distributed parameter systems, Part I, theory, ASME J. Appl. Mech. 60, 1005–1011.

Yang, B., 1994. Distributed transfer function analysis of complex distributed systems. ASME J. Appl. Mech. 61, 84–92.

Yang, B., Tan, C.A., 1992. Transfer functions of one-dimensional distributed parameter system. AIAA Journal 59, 1009–1014.

Yang, B., Zhou, J., 1995. Analysis of ring-stiffened cylindrical shells. ASME J. Appl. Mech. 33, 1005–1014.

Yang, B., Zhou, J., 1996. Semi-analytic solution of 2-D elasticity problems by the strip distributed transfer function method. Int. J. Solids Structures 33, 3983–4005.

Zhou, J., Yang, B., 1995. Distributed transfer function method for analysis of cylindrical shells. AIAA J. 33, 1698–1708.

Zhou, J., Yang, B., 1996. Strip distributed transfer function method for analysis of plates. Int. J. Num. Meth. Engng 39, 1915–1932.